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Exact finite lattice results for Z_n symmetric spin models

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Abstract. We obtain the exact finite lattice partition functions for a sequence of two-dimensional Z_n symmetric spin models, interpolating between the two-phase ferromagnetic Potts model and the three-phase ($n>4$) vector model. We present these results through the distribution of zeros of the partition function in the complex coupling variable. We find that the phase structure for both two- and three-phase models is manifested in this presentation. In the Potts model case the zeros approaching the ferromagnetic phase transition point are restricted to a circular locus (although well away from the ferromagnetic region this restriction does not apply). As the interpolation proceeds the simple ferromagnetic locus is modified, revealing a three-phase structure.

1. Introduction

There has been sustained interest in the phase structure of Z_n symmetric models in two dimensions [1–5]. Most recently, for example, it has been suggested that the crossover point between two- and three-phase behaviour ($n>4$) may be associated with solvable submanifolds of the Andrews–Baxter–Forrester model [5, 6]. At the same time there has been mounting interest in finite or semi-infinite lattice probes of the global analytic structure [7] of the free energy for statistical mechanical models in general [8–10] (in particular resulting in a solution of the q-colouring problem on the triangular lattice [11]).

In the present paper we give a brief report of the results of applying such probes to Z_n symmetric models. We find that the distribution of zeros of the partition function in the complex exponentiated coupling variable [8, 9] manifests the phase structure in both two and three-phase regions. We are thus able to use this technique to explore the crossover region in the cases $n=5$ and 6.

The Z_n symmetric spin model partition function Z may be written [1]

$$Z(\beta, \beta_1, \ldots, \beta_{\tau(n)}) = \sum_{\substack{\text{configurations}\\ \{p_i\}}} \exp\Bigg(\beta \sum_{\substack{\text{lattice}\\ \text{links}\\ ij}} \chi_{\{\beta_r\}}(p_i - p_j)\Bigg) \tag{1}$$

where

$$\chi_{\{\beta_r\}}(p) = \sum_{r=1,\tau(n)} \beta_r \cos(2\pi r p/n) \tag{2}$$

$$\tau(n) = \begin{cases} \frac{1}{2}n & \text{for } n \text{ even} \\ \frac{1}{2}(n-1) & \text{for } n \text{ odd} \end{cases}$$

and the spin variables $S_j = \exp[2\pi i p_j/n)$ where $p_j \in \{0, \ldots, n-1\}$ on the sites j of a (square) lattice.

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To obtain a single-parameter model within this scheme for $n > 4$ we fix the parameters $\{\beta_r\}$ by specifying the value of $\chi_{\{\beta_r\}}(p)$ for each $p \in \{0, \ldots, \tau(n)\}$. Thus $\chi \equiv \{\chi(0), \ldots, \chi(\tau(n))\} = \{1, 0, 0, \ldots\}$ gives the (two-phase) Potts model; $\chi(p) = \cos(2\pi p/n)$ gives the (three-phase) vector model, and so on.

Now, independently of χ, configurations in (1) may be conveniently represented by the boundaries of areas of aligned spins (see [1] for details). Briefly, for each pair of adjacent spins i and j, $|p_i - p_j|$ bits of directed boundary or 'string' are drawn perpendicularly across the corresponding link. When this is done the strings all either close, or end at 'vortices' (sources (sinks) for n strings). These objects all occur in the sum over configurations (i.e. the entropy is model independent) but different specifications for χ assign them different energies.

We may then use the arguments of Einhorn *et al* [1] to review the phase structure of these models. At low temperature (high β) the Boltzmann factors for mostly aligned configurations are largest, outweighing the rarity of such configurations in the sum. As the temperature rises the energy penalty for configurations with strings becomes lower and they condense, driving an order–disorder transition. However, in the vector model overlapping strings are energetically penalised so it is primarily single-string loops which appear. Einhorn *et al* have shown that vortex-antivortex pairs then become significant, and the subsequent appearance of unbound vortices drives another transition to a fully disordered phase. By contrast, in the Potts model no more energy is lost when two strings cross the same link, so multistring loops appear at the first transition, completely disordering the system.

The significance of vortices thus depends on the relationship between energies for different relative alignments. For example, the Z_5, $\chi \equiv \{3, 1, 0\}$ model requires less energy to have two strings together than separated. Thus multistring loops are favoured over vortices, suggesting a Potts-like structure. This argument suggests that the borderline between two- and three-phase models will occur in models requiring roughly equal amounts of energy for separate or overlapping strings (e.g. $\chi \equiv \{2, 1, 0\}$).

Exact results for the phase transition points are not available to test this picture (see [1, 12, 13]). Finite lattice results tend to smooth out the effect of a transition, making it difficult to differentiate between one and two nearby transitions. However, the distribution of zeros of Z in complex $\exp(\beta)$ can be used to overcome this difficulty, even on a finite lattice.

The connection between energy/entropy arguments and these zeros close to the real axis is as follows. For $\exp(-\beta) = r\exp(i\phi)$ with $r < 1$ and small ϕ, terms in Z corresponding to configurations with low string density (typical energy $\sim E$, say) have low entropy but large magnitude Boltzmann factors. Since ϕ is small these terms add more or less coherently. Those corresponding to high string density, say (energy$\sim E + E'$) have high entropy but small Boltzmann factor and add coherently within themselves, but for $\phi \sim [(2n+1)/E']\pi$ (integer n) may cancel with low string terms. If the balance of energy and entropy is correct then $Z = 0$. Similar arguments would cover the vortex unbinding transition. At large ϕ the local coherence of contributions is lost and this perspective is obscured, but for small ϕ we see that the presence of zeros can indicate the appearance of phase transitions on the real axis in the thermodynamic limit (i.e. as $E' \to \infty$). For full details see [8, 9, 14].

For the Potts model we know (see [9]) that zeros extend for some distance into the complex plane in a simple locus given by the inversion circle of the duality transformation (the unit circle in the variable $Y = [\exp(\beta) - 1]/\sqrt{n}$). Indeed, for $n \geq 4$, Hintermann *et al* [15] proved that zeros are restricted to this circle in a neighbourhood of β positive. Clearly such a simple picture cannot apply in general, when multiple phase transitions appear. We can now use the Z_n models to examine the modification in the distribution of zeros as χ is altered from the Potts model to the vector model.

2. Results

The simplest cases from a calculational viewpoint are $n = 5, 6$. Our examples are chosen from the continuum of possible specifications of χ so as to allow plotting of zeros in $\exp(-\beta)$ (i.e. integer entries in χ). It is easy to see using the Newton–Raphson method that such examples are adequately representative of Z_n models in general, i.e. at least in the region of interest close to the positive real axis. Figure 1 shows the distribution of zeros in $\exp(-\beta)$ for the 7×9 lattice five-state Potts model. The boundary conditions (vectors in the transfer matrix formalism) satisfy requirements, discussed in [14], that they be orthogonal to all transfer matrix eigenvectors except those associated with the branch partners [7] of the largest eigenvalue in the physical region. We note the part circle of zeros pinching the positive real axis at the order–disorder transition point. Comparing this with figure 2, which shows the Z_s, $\chi \equiv \{3, 1, 0\}$ model on a 6×7 lattice, we confirm the string-vortex prediction that this model also has one transition. However, the line of zeros falls on no obvious locus, corresponding to the loss of self-duality. In this and subsequent models the duality transformation [13] does not preserve χ except at a finite set of β values. In this case there is just a single self-dual value $\beta \approx 0.62$, cf the two zeros closest to the real axis in the figure at $\beta \approx (0.61 \pm 0.06i)$. This agreement is typical of models in the two-phase region with a unique self-dual point.

It is an unavoidable restriction of integer valued χ that the eigenvalues of the finite lattice transfer matrix are algebraic functions of $\exp(-\beta)$, so in this sense our results are always approximations to algebraic curves [7, 9]. However, it is possible to show, again using the Newton method, that they also give the approximate location of the

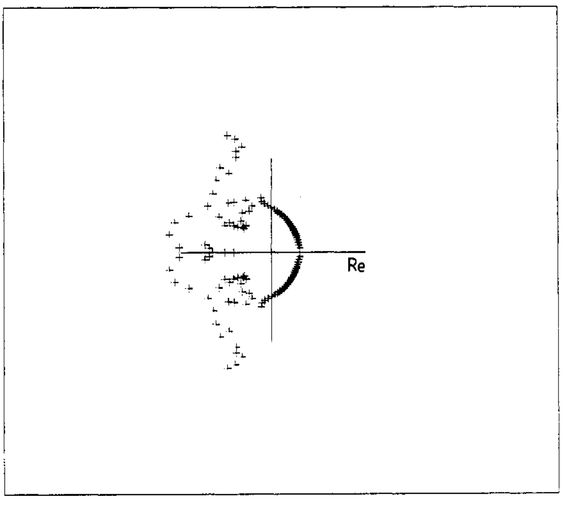

Figure 1. Zeros of the partition function in the complex $\exp(-\beta)$ plane for the 7×9 lattice, $n = 5$, $\chi \equiv \{1, 0, 0\}$ or Potts model. In this and subsequent figures the scale is set by unit length of the positive real axis.

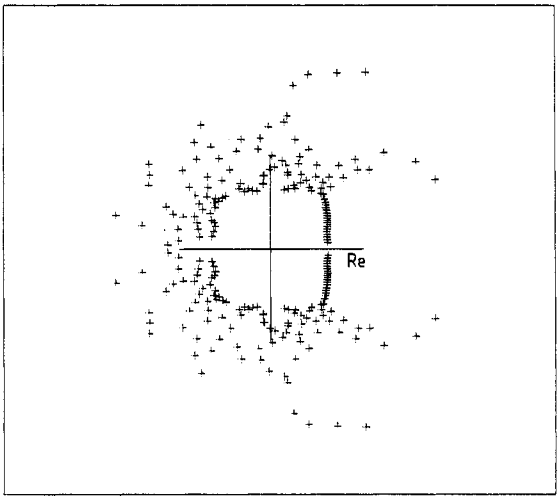

Figure 2. Zeros of the partition function for the 6×7 lattices, $n=5$, $\chi\equiv\{3,1,0\}$ model.

more general curves in complex β associated with other χ values. By restricting attention to lattices of roughly equal size in each direction we prevent the swamping of the finite lattice image by algebraic curves. These may, in any case, disappear in the full thermodynamic limit, since they may become arbitrarily branched or convoluted (although this does not happen in the Potts case). For the above reasons our analysis of these models is essentially limited to inspection of the figures†. If we believe that we can distinguish an algebraic curve on *this* evidence then the line density of zeros in such cases will give an approximation to the specific heat critical exponent [16] (see later).

Figure 3 shows $\chi\equiv\{2,1,0\}$ on a 7×9 lattice. We see that the line of zeros has broadened into a band corresponding to a narrow range of β values in the intermediate phase. Figure 4 then shows that in the $\chi\equiv\{3,2,0\}$ model (here using a 6×7 lattice) this band has developed into two lines. These lines do not have the coherent appearance of the Potts line, and indeed we find that the magnitude of Z between the lines in the complex plane is very small compared to that outside. This suggests that the thermodynamic limit Z may have a ribbon of zeros in this region (i.e. the region is filled with free energy branch cuts [17]) with greatest density at the edges, which is consistent with the algebraic decay of correlations in this region (see [1]). Note that $\chi\equiv\{3,2,0\}$ (i.e. $\beta_1/\beta_2\approx-18$) is a good approximation to the vector model.

A similar sequence occurs on interpolating between Potts and vector models with other n values. For example, figure 5 shows that the Z_6 cubic model [2] $\chi\equiv\{2,1,1,0\}$

† As we have suggested, our calculations (involving sums over an *enormous* number of configurations) are made possible by an algebraic transfer matrix technique (see [9]). Obtaining the largest eigenvalue of the transfer matrix is like raising it to a high power (again, see [9]), i.e. making the lattice very long in the layering direction. Computationally, this is a relatively straightforward process. What we are saying is that, in the present context, it is misleading as far as the thermodynamic limit is concerned unless accompanied by an increase in the transverse size. Unfortunately we are at the current practical limit of this dimension. In short, it is expedient for our purposes to keep the lattice roughly square. The method of analysis is then to present the distributions of zeros (obtained by a high precision version of the Newton technique from the various partition functions) in graphic form and search for patterns by eye (see also [11])!

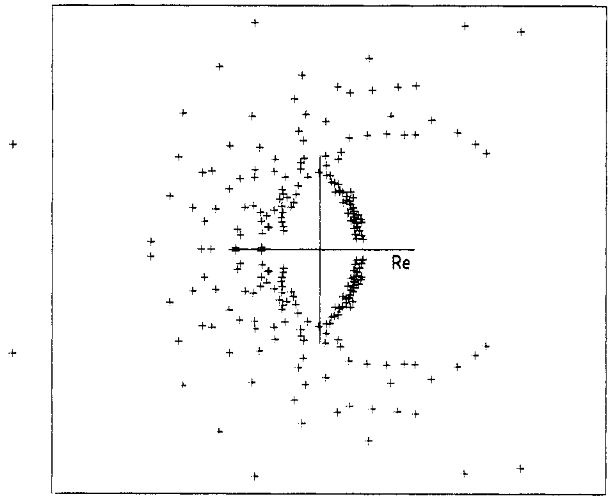

Figure 3. Zeros for the 7×9 lattice, $n = 5$, $\chi \equiv \{2, 1, 0\}$ model.

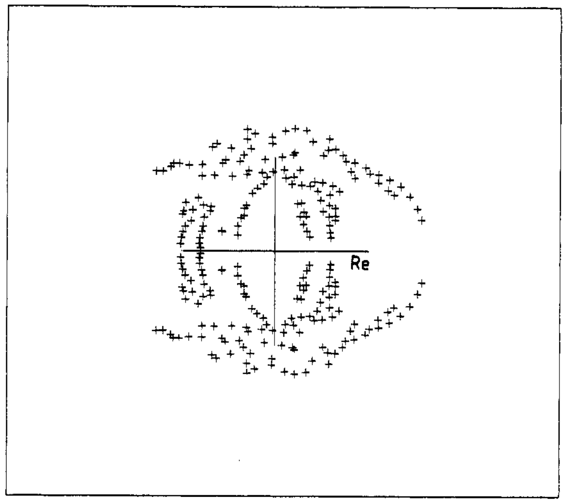

Figure 4. Zeros for the 6×7 lattice, $n = 5$, $\chi \equiv \{3, 2, 0\}$ model.

(using a 6×7 lattice) has a single transition, while figures 6 and 7 ($\chi \equiv \{3, 2, 1, 0\}$ and the vector model $\chi \equiv \{4, 3, 1, 0\}$) confirm a broadening intermediate phase.

If we normalise these models to a common energy for aligned spins then the low-temperature transition 'appears' and moves to lower temperature as the single-string favourability (see above) is increased, while the other transition remains fixed. This is consistent with the string–vortex argument.

As has previously been discussed [8, 16] it is possible to obtain an estimate of the specific heat critical exponent of a phase transition from the line separation of zeros approaching the phase transition point. This presupposes a one-dimensional distribution. Where such a distribution occurs the results are consistent with a first-order transition. Elsewhere size effects prevent a convincing analysis (see also [18]). However, there is pictorial evidence (compare figures 2 and 3, and 5 and 6) to suggest

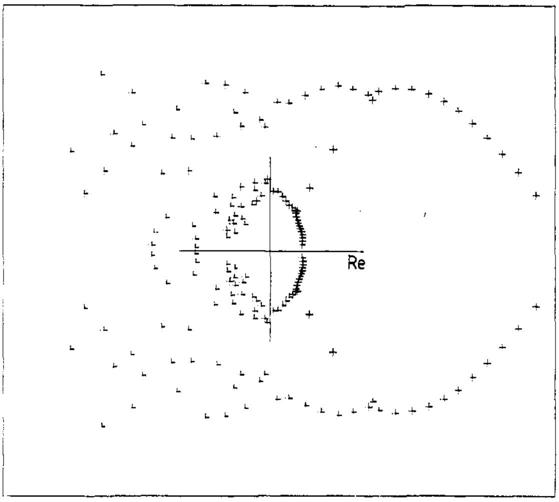

Figure 5. Zeros for the 6×7 lattice, $n=6$, $\chi\equiv\{2,1,1,0\}$ model.

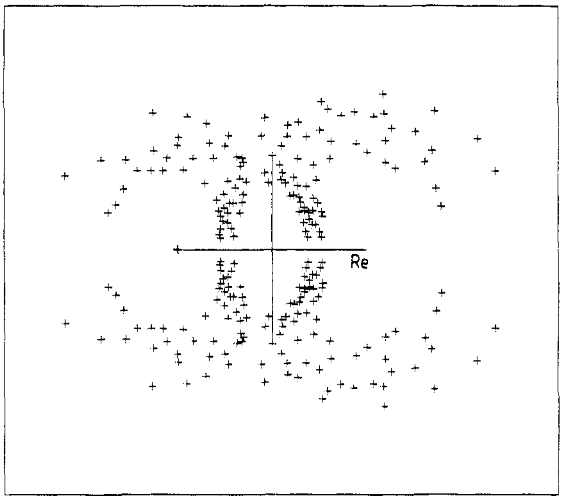

Figure 6. Zeros for the 6×7 lattice, $n=6$, $\chi\equiv\{3,2,1,0\}$ model.

that the algebraic (one-dimensional) distribution of zeros may break down close to the crossover point in these models. This could, in principle, allow the appearance of a single second-order transition [5] at the crossover point.

3. Summary

We have shown that the phase structure of three-phase models is manifested in the zero distribution of the partition function. We have thus confirmed that the string-vortex description of Z_n models gives a rough guide to the point of crossover between two and three phases. We find evidence that on the two-phase side the transition is first order and (at least for $n=5$) occurs at the self-dual point.

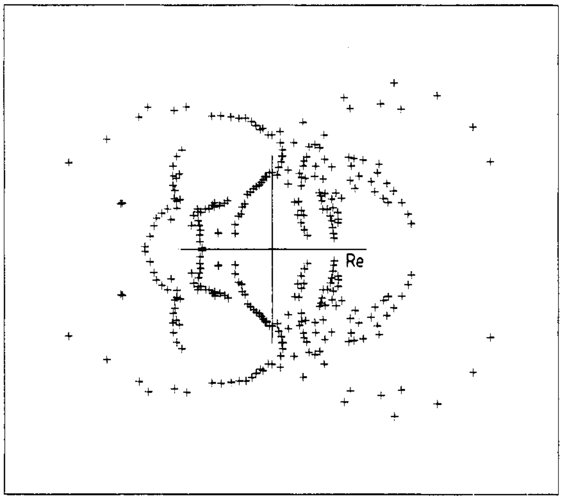

Figure 7. Zeros for the 6×7 lattice, $n = 6$, $\chi \equiv \{4, 3, 1, 0\}$ or vector model.

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